	A Fast Adaptive Multipole Algorithm in Three Dimensions	An overview of fast multipole method	Validation of Vortex Methods as a Direct Numerical Simulation of Turbulence
spherical harmonic	$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x).$ $Y_n^m(\theta,\phi) = \sqrt{\frac{(n-\lvert m\rvert)!}{(n+\lvert m\rvert)!}} \cdot P_n^{\lvert m\rvert}(\cos\theta)e^{im\phi}.$	$P_n^m(u) = (1-u^2)^{\frac{m}{2}}\frac{\partial^m}{\partial u^m}P_n(u)$ $Y_n^m(y) = (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^{\lvert m\rvert}(\cos\theta_y)e^{im\phi_y}$	$Y_n^m(\theta,\phi) = \sqrt{\frac{(n-\lvert m\rvert)!}{(n+\lvert m\rvert)!}}P_n^{\lvert m\rvert}(cos\theta)e^{im\phi}.$
P2M	$\Phi(X) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{M_n^m}{r^{n+1}} \cdot Y_n^m(\theta,\phi),$ $M_n^m = \sum_{i=1}^{N} q_i \cdot \rho_i^n \cdot Y_n^{-m}(\alpha_i,\beta_i).$	$\hat{f}_p(y) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} M_n^m\frac{Y_n^m(y)}{\lvert y\rvert^{n+1}}$ $M_n^m = \sum_i q_i\lvert x_i\rvert^n Y_n^{-m}(x_i)$	$\Phi(\mathbf{x}_i) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} r_i^{-n-1}Y_n^m(\theta_i,\phi_i)\underbrace{\left\{\sum_j^N \mathbf{q}_j\rho_j^n Y_n^{-m}(\alpha_j,\beta_j)\right\}}_{M_n^m}$
M2M	$M_j^k = \sum_{n=0}^{j}\sum_{m=-n}^{n}\frac{O_{j-n}^{k-m} \cdot i^{\lvert k\rvert-\lvert m\rvert-\lvert k-m\rvert} \cdot A_n^m \cdot A_{j-n}^{k-m} \cdot \rho^n \cdot Y_n^{-m}(\alpha,\beta)}{A_j^k},$	$\tilde{M}_n^m = \sum_{j=0}^{n}\sum_{k=-j}^{j} M_{n-j}^{m-k}\frac{i^{\lvert m\rvert}}{i^{\lvert k\rvert}i^{\lvert m-k\rvert}}\frac{A_j^k A_{n-j}^{m-k}}{A_n^m}\lvert z\rvert^j Y_j^k(z)$	$M_j^k = \sum_{n=0}^{j}\sum_{m=-n}^{n}\frac{\hat{M}_{j-n}^{k-m} i^{\lvert k\rvert-\lvert m\rvert-\lvert k-m\rvert}A_n^m A_{j-n}^{k-m}\rho^n Y_n^{-m}(\alpha,\beta)}{(-1)^n A_j^k}$
M2L	$L_j^k = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{O_n^m \cdot i^{\lvert k-m\rvert-\lvert k\rvert-\lvert m\rvert} \cdot A_n^m \cdot A_j^k \cdot Y_{j+n}^{m-k}(\alpha,\beta)}{(-1)^n A_{j+n}^{m-k} \cdot \rho^{j+n+1}}$	$L_n^m = \sum_{j=0}^{\infty}\sum_{k=-j}^{j}\frac{M_j^k}{(-1)^j}\frac{i^{\lvert m-k\rvert}}{i^{\lvert k\rvert}i^{\lvert m\rvert}}\frac{A_j^k A_n^m}{A_{j-n}^{k-m}}\frac{Y_{j+n}^{k-m}(z)}{\lvert z\rvert^{j+n+1}}$	$L_j^k = \sum_{n=0}^{p-1}\sum_{m=-n}^{n}\frac{M_n^m i^{\lvert k-m\rvert-\lvert k\rvert-\lvert m\rvert}A_n^m A_j^k Y_{j+n}^{m-k}(\alpha,\beta)}{(-1)^{j+k}A_{j+n}^{m-k}\rho^{j+n+1}},$
L2L	$L_j^k = \sum_{n=j}^{p}\sum_{m=-n}^{n}\frac{O_n^m \cdot i^{\lvert m\rvert-\lvert m-k\rvert-\lvert k\rvert} \cdot A_{n-j}^{m-k} \cdot A_j^k \cdot Y_{n-j}^{m-k}(\alpha,\beta) \cdot \rho^{n-j}}{(-1)^{n+j} \cdot A_n^m},$	$\tilde{L}_n^m = \sum_{j=n}^{p-1}\sum_{k=-j}^{j}\frac{L_j^k}{(-1)^{j+n}}\frac{i^{\lvert k\rvert}}{i^{\lvert k-m\rvert}i^{\lvert m\rvert}}\frac{A_{j-n}^{k-m}A_n^m}{A_j^k}Y_{j-n}^{k-m}(-z)\lvert z\rvert^{j-n}$	$L_j^k = \sum_{n=j}^{p-1}\sum_{m=-n}^{n}\frac{\hat{L}_n^m i^{\lvert m\rvert-\lvert k\rvert-\lvert m-k\rvert}A_{n-j}^{m-k}A_j^k\rho^{n-j}Y_{n-j}^{m-k}(\alpha,\beta)}{A_n^m}$
L2P	$\Phi(X) = \sum_{j=0}^{\infty}\sum_{k=-j}^{j} L_j^k \cdot Y_j^k(\theta,\phi) \cdot r^j$	$\hat{f}^*(y) = \sum_n\sum_{m=-n}^{n} L_n^m\lvert z-y\rvert^n Y_n^m(z-y)$	$\Phi(\mathbf{x}_i) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} r_i^n Y_n^m(\theta_i,\phi_i)\underbrace{\left\{\sum_j^N \mathbf{q}_j\rho_j^{-n-1}Y_n^{-m}(\alpha_j,\beta_j)\right\}}_{L_n^m}$
	$A_n^m = \frac{(-1)^n}{\sqrt{(n-m)! \cdot (n+m)!}}.$	$A_n^m = \frac{(-1)^n}{\sqrt{(n-m)!(n+m)!}}.$	$A_n^m = \frac{(-1)^n}{(n-m)!(n+m)!}.$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

in spherical coordinate

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} = 0.$$

assume: $f(r,\theta,\varphi) = R(r)\,Y(\theta,\varphi)$

$$\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{R}{r^2 \sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = -\frac{1}{Y\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) - \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}$$

$$\begin{cases} \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \lambda, \\ \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2} = -\lambda. \end{cases}$$

assume: $Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi)$

$$\frac{\Phi}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -\lambda\Theta\Phi$$

$$\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\lambda\sin^2\theta$$

$$\frac{sin\theta}{\Theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$$

$$\begin{cases} \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2 \\ \lambda\sin^2\theta + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2 \end{cases}$$

$$\Phi(\varphi) = \cos(m\varphi) + i sin(m\varphi) = e^{im\varphi}$$

$m = 0, \pm1, \pm2, \ldots$

$let: x = cos\theta$

$$(1-x^2)\frac{d^2\theta}{dx^2} + \left(\lambda - \frac{m^2}{1-x^2}\right)\Theta = 0$$

$let: \lambda = l(l+1)$

$$(1-x^2)\frac{d^2\theta}{dx^2} + \left(l(l+1) - \frac{m^2}{1-x^2}\right)\Theta = 0$$

$only\ when$
$l = 0,1,2,\ldots$
$m \in [0, l]$

$$\Theta(\theta) = P_l^m(cos\theta)$$

associated legendre polynomials

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x),$$

sometimes omitted

move to here

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\phi}.$$

normalization factor[3]4.7

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell.$$

$allows\ m\ in\ [-l, l]$

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x).$$

about exafmm/mininal code

https://github.com/exafmm/exafmm

$$\widehat{M}_n^m = \sum_{i=1}^{N} q_i \cdot \rho_i^n \cdot Y_n^{-m}(\alpha_i, \beta_i) \Rightarrow \widehat{M}_n^m = \sum_{i=1}^{N} q_i \cdot \rho_i^n \cdot Y_n^{-m}(\alpha_i, \beta_i) \Rightarrow \widehat{M}_n^m = \sum_{i=1}^{N} q_i \cdot \underbrace{\left(\rho_i^n \cdot A_n^m \cdot Y_n^{-m}(\alpha_i, \beta_i)\right)}_{\tilde{Y}_n^m}$$

insert

$$M_j^k = \sum_{n=0}^{j} \sum_{m=-n}^{n} \frac{\widehat{M}_{j-n}^{k-m} \cdot i^{|k|-|m|-|k-m|} \cdot A_n^m \cdot A_{j-n}^{k-m} \cdot \rho^n \cdot Y_n^{-m}(\alpha, \beta)}{A_j^k} \Rightarrow M_j^k = \frac{1}{A_j^k} \sum_{n=0}^{j} \sum_{m=-n}^{n} \left(\widehat{M}_{j-n}^{k-m} \cdot A_{j-n}^{k-m}\right) \cdot i^{|k|-|m|-|k-m|} \cdot \left(\rho^n \cdot A_n^m \cdot Y_n^{-m}(\alpha, \beta)\right) \Rightarrow M_j^k = \sum_{n=0}^{j} \sum_{m=-n}^{n} \widehat{M}_{j-n}^{k-m} \cdot i^{|k|-|m|-|k-m|} \cdot \underbrace{\left(\rho^n \cdot A_n^m \cdot Y_n^{-m}(\alpha, \beta)\right)}_{\tilde{Y}_n^m}$$

cancel out

$$L_j^k = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{M_n^m \cdot i^{|k-m|-|k|-|m|} \cdot A_n^m \cdot A_j^k \cdot Y_{j+n}^{m-k}(\alpha, \beta)}{(-1)^n A_{j+n}^{m-k} \cdot \rho^{j+n+1}} \Rightarrow L_j^k = A_j^k \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{(M_n^m \cdot A_n^m) \cdot i^{|k-m|-|k|-|m|} \cdot Y_{j+n}^{m-k}(\alpha, \beta)}{(-1)^n A_{j+n}^{m-k} \cdot \rho^{j+n+1}} \Rightarrow L_j^k = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{M_n^m \cdot i^{|k-m|-|k|-|m|} \cdot Y_{j+n}^{m-k}(\alpha, \beta)}{(-1)^n A_{j+n}^{m-k} \cdot \rho^{j+n+1}} \quad \check{Y}_n^m$$

cancel out

$$\hat{L}_j^k = \sum_{n=j}^{p-1} \sum_{m=-n}^{n} \frac{L_n^m \cdot i^{|m|-|m-k|-|k|} \cdot A_{n-j}^{m-k} \cdot A_j^k \cdot Y_{n-j}^{m-k}(\alpha, \beta) \cdot \rho^{n-j}}{(-1)^{n+j} \cdot A_n^m} \Rightarrow \hat{L}_j^k = A_j^k \sum_{n=j}^{p-1} \sum_{m=-n}^{n} \frac{L_n^m \cdot i^{|m|-|m-k|-|k|} \cdot \left(A_{n-j}^{m-k} \cdot Y_{n-j}^{m-k}(\alpha, \beta) \cdot \rho^{n-j}\right)}{(-1)^{n+j} \cdot A_n^m} \Rightarrow \hat{L}_j^k = \sum_{n=j}^{p-1} \sum_{m=-n}^{n} \frac{L_n^m \cdot i^{|m|-|m-k|-|k|} \cdot \underbrace{\left(A_{n-j}^{m-k} \cdot Y_{n-j}^{m-k}(\alpha, \beta) \cdot \rho^{n-j}\right)}_{\tilde{Y}_{n-j}^{m-k}}}{(-1)^{n+j}}$$

insert

$$\Phi(X) = \sum_{j=0}^{p-1} \sum_{k=-j}^{j} \hat{L}_j^k \cdot Y_j^k(\theta, \emptyset) \cdot r^j \Rightarrow \Phi(X) = \sum_{j=0}^{p-1} \sum_{k=-j}^{j} \hat{L}_j^k \cdot Y_j^k(\theta, \emptyset) \cdot r^j \Rightarrow \Phi(X) = \sum_{j=0}^{p-1} \sum_{k=-j}^{j} \hat{L}_j^k \cdot \underbrace{\left(A_j^k \cdot Y_j^k(\theta, \emptyset) \cdot r^j\right)}_{\tilde{Y}_j^k}$$

$$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

$$Y_n^m(\theta, \emptyset) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(cos\theta) e^{im\emptyset}$$

$$A_n^m = \frac{(-1)^n}{\sqrt{(n-m)! \cdot (n+m)!}}.$$

$$\tilde{Y}_n^m(\theta, \emptyset) = \frac{(-1)^n}{(n+|m|)!} \cdot P_n^{|m|}(cos\theta) e^{im\emptyset} \rho^n$$

$$\check{Y}_n^m(\theta, \emptyset) = (-1)^n (n-|m|)! \cdot P_n^{|m|}(cos\theta) e^{im\emptyset} \rho^{-n-1}$$

$$\hat{M}_n^m = \sum_{i=1}^{N} q_i \cdot \left(\rho_i^n \cdot A_n^m \cdot Y_n^{-m}(\alpha_i, \beta_i)\right)$$

$$M_j^k = \sum_{n=0}^{j} \sum_{m=-n}^{n} \hat{M}_{j-n}^{k-m} \cdot i^{|k|-|m|-|k-m|} \cdot \left(\rho^n \cdot A_n^m \cdot Y_n^{-m}(\alpha, \beta)\right)$$

$$L_j^k = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{M_n^m \cdot i^{|k-m|-|k|-|m|} \cdot Y_{j+n}^{m-k}(\alpha, \beta)}{(-1)^n A_{j+n}^{m-k} \cdot \rho^{j+n+1}}$$

$$\hat{L}_j^k = \sum_{n=j}^{p-1} \sum_{m=-n}^{n} \frac{L_n^m \cdot i^{|m|-|m-k|-|k|} \cdot \left(A_{n-j}^{m-k} \cdot Y_{n-j}^{m-k}(\alpha, \beta) \cdot \rho^{n-j}\right)}{(-1)^{n+j}}$$

$$\Phi(X) = \sum_{j=0}^{p-1} \sum_{k=-j}^{j} \hat{L}_j^k \cdot \left(A_j^k \cdot Y_j^k(\theta, \emptyset) \cdot r^j\right)$$

Parity

$$Y_{lm}(\pi - \theta, \pi + \varphi) = (-1)^l Y_{lm}(\theta, \varphi)$$

$$\hat{M}_n^m = \sum_{i=1}^{N} q_i \cdot \left(\rho_i^n \cdot A_n^m \cdot Y_n^{-m}(\alpha_i, \beta_i)\right) \qquad M_n^m = \overline{M_n^m}$$

$$M_j^k = \sum_{n=0}^{j} \sum_{m=-n}^{n} \hat{M}_{j-n}^{k-m} \cdot \frac{i^{|k|-|m|-|k-m|}}{(-1)^n} \cdot \left(\rho^n \cdot A_n^m \cdot Y_n^{-m}(\alpha, \beta)\right) \qquad M_j^k = \overline{M_j^k}$$

$$L_j^k = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{M_n^m \cdot i^{|k-m|-|k|-|m|} \cdot Y_{j+n}^{m-k}(\alpha, \beta)}{(-1)^j A_{j+n}^{m-k} \cdot \rho^{j+n+1}} \qquad L_j^k = \overline{L_j^k}$$

$$\hat{L}_j^k = \sum_{n=j}^{p-1} \sum_{m=-n}^{n} L_n^m \cdot i^{|m|-|m-k|-|k|} \cdot \left(A_{n-j}^{m-k} \cdot Y_{n-j}^{m-k}(\alpha, \beta) \cdot \rho^{n-j}\right) \qquad L_j^k = \overline{L_j^k}$$

$$\Phi(X) = \sum_{j=0}^{p-1} \sum_{k=-j}^{j} \hat{L}_j^k \cdot \left(A_j^k \cdot Y_j^k(\theta, \emptyset) \cdot r^j\right)$$

$$output : \tilde{Y}_n^m(\theta,\emptyset) = \rho^n \cdot A_n^m \cdot Y_n^m(\alpha,\beta)$$

```
void evalMultipole(real_t rho, real_t alpha, real_t beta, complex_t * Ynm, complex_t * YnmTheta) {
  real_t x = std::cos(alpha);                                    // x = cos(alpha)
  real_t y = std::sin(alpha);                                    // y = sin(alpha)
  real_t invY = y == 0 ? 0 : 1 / y;                              // 1 / y
  real_t fact = 1;                                               // Initialize 2 * m + 1
  real_t pn = 1;                                                 // Initialize Legendre polynomial Pn
  real_t rhom = 1;                                               // Initialize rho^m
  complex_t ei = std::exp(I * beta);                             // exp(i * beta)
  complex_t eim = 1.0;                                           // Initialize exp(i * m * beta)
  for (int m=0; m<P; m++) {                                      // Loop over m in Ynm
    real_t p = pn;                                               //  Associated Legendre polynomial Pnm
    int npn = m * m + 2 * m;                                     //  Index of Ynm for m > 0
    int nmn = m * m;                                             //  Index of Ynm for m < 0
    Ynm[npn] = rhom * p * eim;                                   //  rho^m * Ynm for m > 0
    Ynm[nmn] = std::conj(Ynm[npn]);                              //  Use conjugate relation for m < 0
    real_t p1 = p;                                               //  Pnm-1
    p = x * (2 * m + 1) * p1;                                    //  Pnm using recurrence relation
    YnmTheta[npn] = rhom * (p - (m + 1) * x * p1) * invY * eim; //  theta derivative of r^n * Ynm
    rhom *= rho;                                                 //  rho^m
    real_t rhon = rhom;                                          //  rho^n
    for (int n=m+1; n<P; n++) {                                  //  Loop over n in Ynm
      int npm = n * n + n + m;                                   //   Index of Ynm for m > 0
      int nmm = n * n + n - m;                                   //   Index of Ynm for m < 0
      rhon /= -(n + m);                                          //   Update factorial
      Ynm[npm] = rhon * p * eim;                                 //   rho^n * Ynm
      Ynm[nmm] = std::conj(Ynm[npm]);                            //   Use conjugate relation for m < 0
      real_t p2 = p1;                                            //   Pnm-2
      p1 = p;                                                    //   Pnm-1
      p = (x * (2 * n + 1) * p1 - (n + m) * p2) / (n - m + 1);//   Pnm using recurrence relation
      YnmTheta[npm] = rhon * ((n - m + 1) * p - (n + 1) * x * p1) * invY * eim;// theta derivative
      rhon *= rho;                                               //   Update rho^n
    }                                                            //  End loop over n in Ynm
    rhom /= -(2 * m + 2) * (2 * m + 1);                          //  Update factorial
    pn = -pn * fact * y;                                         //  Pn
    fact += 2;                                                   //  2 * m + 1
    eim *= ei;                                                   //  Update exp(i * m * beta)
  }                                                              // End loop over m in Ynm
}
```

$\tilde{Y}_0^0, \tilde{Y}_1^1, \tilde{Y}_2^2, \tilde{Y}_3^3$ (Ynm[npn])

$\tilde{Y}_0^0, \tilde{Y}_1^{-1}, \tilde{Y}_2^{-2}, \tilde{Y}_3^{-3}$ (Ynm[nmn])

$(\tilde{Y}_1^0, \tilde{Y}_2^0, \tilde{Y}_3^0)(\tilde{Y}_2^1, \tilde{Y}_3^1)(\tilde{Y}_3^2)$ (Ynm[npm])

$(\tilde{Y}_1^0, \tilde{Y}_2^0, \tilde{Y}_3^0)(\tilde{Y}_2^{-1}, \tilde{Y}_3^{-1})(\tilde{Y}_3^{-2})$ (Ynm[nmm])

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

$$A_n^m = \frac{(-1)^n}{\sqrt{(n-m)! \cdot (n+m)!}}.$$

$$Y_n^m(\theta,\emptyset) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(cos\theta) e^{im\emptyset}$$

$$Y_n^{-m}(\theta,\emptyset) = \sqrt{\frac{(n-|-m|)!}{(n+|-m|)!}} \cdot P_n^{|-m|}(cos\theta) e^{-im\emptyset} = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(cos\theta) e^{-im\emptyset} = \overline{Y_n^m(\theta,\emptyset)}$$

if $0 \le m \le n$:

$$\tilde{Y}_n^m(\theta,\emptyset) = \rho^n \cdot A_n^m \cdot Y_n^m(\theta,\emptyset) = (-1)^n \frac{1}{(n+m)!} \cdot P_n^m(cos\theta) e^{im\emptyset} \cdot \rho^n$$

else if $-n \le m < 0$:

$$\tilde{Y}_n^{-m}(\theta,\emptyset) = \rho^n \cdot A_n^{-m} \cdot Y_n^{-m}(\theta,\emptyset) = (-1)^n \frac{1}{(n+m)!} \cdot P_n^{|m|}(cos\theta) e^{-im\emptyset} \cdot \rho^n = \overline{\tilde{Y}_n^m(\theta,\emptyset)}$$

$pn : P_m^m(cos\theta)$ → $P_{m+1}^{m+1} = -(2m+1)\sqrt{1-x^2} P_m^m$

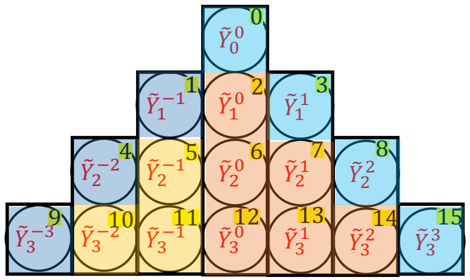
$p : P_n^m(cos\theta)$ → $(n-m+1) P_n^m = (2n+1) x P_{n-1}^m - (n+m) P_{n-2}^m$

$rhom : (-1)^m \frac{\rho^m}{(m+m)!}$

$rhon : (-1)^n \frac{\rho^n}{(n+m)!}$

$YnmTheta :$

$$(x^2-1)\frac{d}{dx} P_\ell^m(x) = -(\ell+1) x P_\ell^m(x) + (\ell - m + 1) P_{\ell+1}^m(x)$$



$$output : \tilde{Y}_n^m(\theta, \emptyset) = Y_n^m(\alpha, \beta)/\rho^{n+1} \cdot A_n^m$$

```
void evalLocal(real_t rho, real_t alpha, real_t beta, complex_t * Ynm) {
  real_t x = std::cos(alpha);                          // x = cos(alpha)
  real_t y = std::sin(alpha);                          // y = sin(alpha)
  real_t fact = 1;                                     // Initialize 2 * m + 1
  real_t pn = 1;                                       // Initialize Legendre polynomial Pn
  real_t invR = -1.0 / rho;                            // - 1 / rho
  real_t rhom = -invR;                                 // Initialize rho^(-m-1)
  complex_t ei = std::exp(I * beta);                   // exp(i * beta)
  complex_t eim = 1.0;                                 // Initialize exp(i * m * beta)
  for (int m=0; m<P; m++) {                            // Loop over m in Ynm
    real_t p = pn;                                     //  Associated Legendre polynomial Pnm
    int npn = m * m + 2 * m;                           //  Index of Ynm for m > 0
    int nmn = m * m;                                   //  Index of Ynm for m < 0
    Ynm[npn] = rhom * p * eim;                         //  rho^(-m-1) * Ynm for m > 0
    Ynm[nmn] = std::conj(Ynm[npn]);                    //  Use conjugate relation for m < 0
    real_t p1 = p;                                     //  Pnm-1
    p = x * (2 * m + 1) * p1;                          //  Pnm using recurrence relation
    rhom *= invR;                                      //  rho^(-m-1)
    real_t rhon = rhom;                                //  rho^(-n-1)
    for (int n=m+1; n<P; n++) {                        //  Loop over n in Ynm
      int npm = n * n + n + m;                         //   Index of Ynm for m > 0
      int nmm = n * n + n - m;                         //   Index of Ynm for m < 0
      Ynm[npm] = rhon * p * eim;                       //   rho^n * Ynm for m > 0
      Ynm[nmm] = std::conj(Ynm[npm]);                  //   Use conjugate relation for m < 0
      real_t p2 = p1;                                  //   Pnm-2
      p1 = p;                                          //   Pnm-1
      p = (x * (2 * n + 1) * p1 - (n + m) * p2) / (n - m + 1);//   Pnm using recurrence relation
      rhon *= invR * (n - m + 1);                      //   rho^(-n-1)
    }                                                  //  End loop over n in Ynm
    pn = -pn * fact * y;                               //  Pn
    fact += 2;                                         //  2 * m + 1
    eim *= ei;                                         //  Update exp(i * m * beta)
  }                                                    // End loop over m in Ynm
}
```

- `Ynm[npn]`: $\tilde{Y}_0^0, \tilde{Y}_1^1, \tilde{Y}_2^2, \tilde{Y}_3^3$
- `Ynm[nmn]`: $\tilde{Y}_0^0, \tilde{Y}_1^{-1}, \tilde{Y}_2^{-2}, \tilde{Y}_3^{-3}$
- `Ynm[npm]`: $(\tilde{Y}_1^0, \tilde{Y}_2^0, \tilde{Y}_3^0)(\tilde{Y}_2^1, \tilde{Y}_3^1)(\tilde{Y}_3^2)$
- `Ynm[nmm]`: $(\tilde{Y}_1^0, \tilde{Y}_2^0, \tilde{Y}_3^0)(\tilde{Y}_2^{-1}, \tilde{Y}_3^{-1})(\tilde{Y}_3^{-2})$

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

$$A_n^m = \frac{(-1)^n}{\sqrt{(n-m)! \cdot (n+m)!}}.$$

$$Y_n^m(\theta, \emptyset) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(cos\theta) e^{im\emptyset}$$

$$Y_n^{-m}(\theta, \emptyset) = \sqrt{\frac{(n-|-m|)!}{(n+|-m|)!}} \cdot P_n^{|-m|}(cos\theta) e^{-im\emptyset}$$
$$= \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(cos\theta) e^{-im\emptyset}$$
$$= \overline{Y_n^m(\theta, \emptyset)}$$

$if \;\; 0 \le m \le n$:

$$\tilde{Y}_n^m(\theta, \emptyset) = \rho^n \cdot A_n^m \cdot Y_n^m(\theta, \emptyset) = (-1)^n \frac{1}{(n+m)!} \cdot P_n^m(cos\theta) e^{im\emptyset} \cdot \rho^n$$

$else \; if \;\; -n \le m < 0$:

$$\tilde{Y}_n^{-m}(\theta, \emptyset) = \rho^n \cdot A_n^{-m} \cdot Y_n^{-m}(\theta, \emptyset) = (-1)^n \frac{1}{(n+m)!} \cdot P_n^{|m|}(cos\theta) e^{-im\emptyset} \cdot \rho^n$$
$$= \overline{\tilde{Y}_n^m(\theta, \emptyset)}$$

$pn : \; P_m^m(cos\theta)$ → $P_{m+1}^{m+1} = -(2m+1)\sqrt{1-x^2} P_m^m$

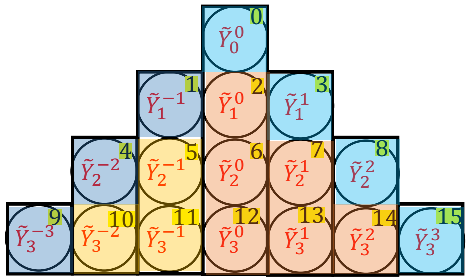
$p : \; P_n^m(cos\theta)$ → $(n-m+1)P_n^m = (2n+1)xP_{n-1}^m - (n+m)P_{n-2}^m$

$rhom : (-1)^m \frac{1}{\rho^{m+1}}$

$rhon : (-1)^n \frac{1}{\rho^{n+1}} (n-m)!$

$YnmTheta :$

$$(x^2-1)\frac{d}{dx}P_\ell^m(x) = -(\ell+1)xP_\ell^m(x) + (\ell-m+1)P_{\ell+1}^m(x)$$



https://mathworld.wolfram.com/SphericalCoordinates.html

$$\frac{\partial}{\partial x} = \cos\theta\sin\phi\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\phi}\,\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\phi}\,\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\phi}{r}\,\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\phi\,\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\,\frac{\partial}{\partial\phi}$$

about definitions of spherical harmonic

A New Version of the Fast Multipole Method for the Laplace Equation in Three Dimensions.

$$f(y) = \sum_{i}^{N} q_i \frac{1}{|y - x_i|}$$

$$Y_n^m(\theta, \phi) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(\cos\theta) e^{im\phi}.$$

$$|y - x_i|^2 = |x_i|^2 + |y|^2 - 2|x_i||y|cos\theta$$

$$\frac{1}{|y - x_i|^2} = \frac{1}{|x_i|^2 + |y|^2 - 2|x_i||y|cos\theta}$$

$$\frac{1}{|y - x_i|} = \frac{1}{|y|}\frac{1}{\sqrt{1 - 2uv + v^2}}, u = cos\theta, v = \frac{|x|}{|y|} \; (v < 1)$$

$$\frac{1}{\sqrt{1 - 2uv + v^2}} = \sum_{n=0}^{\infty} P_n(u) v^n$$

$$\frac{1}{|y - x_i|} = \sum_{n=0}^{\infty} \frac{1}{|y|^{n+1}} P_n(cos\theta)|x|^n$$

$$P_n(u) = P_n(\cos\theta) = \sum_{m=-n}^{n} Y_n^{-m}(x) Y_n^m(y)$$

$$P_n(u) = P_n(\cos\theta) = \sum_{m=-n}^{n} (-1)^m Y_n^{-m}(x) Y_n^m(y)$$

$$\frac{1}{|y - x_i|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |x_i|^n Y_n^{-m}(x_i) \frac{Y_n^m(y)}{|y|^{n+1}}$$

$$\frac{1}{|y - x_i|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m |x_i|^n Y_n^{-m}(x_i) \frac{Y_n^m(y)}{|y|^{n+1}}$$

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M_n^m(multipole expansion)

$$f(y) = \sum_{i}^{N} q_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m |x_i|^n Y_n^{-m}(x_i) \frac{Y_n^m(y)}{|y|^{n+1}}$$
$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m \frac{Y_n^m(y)}{|y|^{n+1}} \boxed{\sum_{i}^{N} q_i |x_i|^n Y_n^{-m}(x_i)}$$

M_n^m(multipole expansion)

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$$\frac{1}{|y - x|} = \frac{1}{|y|}\frac{1}{\sqrt{1 - 2uv + v^2}}, u = cos\theta, v = \frac{|x|}{|y|}$$

$$|x - y|^2 = |y|^2 + |x|^2 - 2|y||x|cos\theta$$

$$\frac{1}{|x - y|^2} = \frac{1}{|y|^2 + |x|^2 - 2|y||x|cos\theta}$$

$$\frac{1}{|x - y|} = \frac{1}{|x|}\frac{1}{\sqrt{1 - 2uv + v^2}}, u = cos\theta, v = \frac{|y|}{|x|}$$

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3 types of definition of spherical harmonics(1 is convention)

type 1: https://en.wikipedia.org/wiki/Spherical_harmonics

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \qquad m \geq 0$$

↓

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

← 不变! →

http://www.jooooow.com/static/pdf/FMM.pdf

$$Y_n^m(\theta,\emptyset) = \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(cos\theta) e^{im\emptyset}$$

$$(n \geq 0, 0 \leq m \leq n)$$

$$Y_n^{-m}(\theta,\emptyset) = \sqrt{\frac{(n+m)!}{(n-m)!}} P_n^{-m}(cos\theta) e^{-im\emptyset}$$

$$= \sqrt{\frac{(n+m)!}{(n-m)!}} (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(cos\theta) e^{-im\emptyset}$$

$$= (-1)^m \sqrt{\frac{(n+m)!}{(n-m)!}} P_n^m(cos\theta) e^{-im\emptyset}$$

$$= (-1)^m \cdot \overline{Y_n^m(\theta,\emptyset)}$$

type 2: https://www.physics.uoguelph.ca/chapter-4-spherical-harmonics

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \qquad m \geq 0$$

↓

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

$$Y_n^m(\theta,\emptyset) = (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(cos\theta) e^{im\emptyset}$$

$$(n \geq 0, 0 \leq m \leq n)$$

$$Y_n^{-m}(\theta,\emptyset) = (-1)^{-m} \sqrt{\frac{(n+m)!}{(n-m)!}} P_n^{-m}(cos\theta) e^{-im\emptyset}$$

$$= (-1)^{-m} \sqrt{\frac{(n+m)!}{(n-m)!}} (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(cos\theta) e^{-im\emptyset}$$

$$= \sqrt{\frac{(n+m)!}{(n-m)!}} P_n^m(cos\theta) e^{-im\emptyset}$$

$$= (-1)^m \overline{Y_n^m(\theta,\emptyset)}$$

type 3:

$$Y_n^m(\theta,\phi) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(\cos\theta) e^{im\phi}.$$

→

$$Y_n^{-m}(\theta,\emptyset) = \sqrt{\frac{(n-|-m|)!}{(n+|-m|)!}} \cdot P_n^{|-m|}(cos\theta) e^{-im\emptyset}$$

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$$Y_n^m(\theta,\emptyset) = \begin{cases} \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(cos\theta) e^{im\emptyset} & ,0 \leq m \leq n \\ (-1)^m \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(cos\theta) e^{im\emptyset} & ,-n \leq m < 0 \end{cases}$$

$$Y_n^{-m}(\theta,\emptyset) = (-1)^m \cdot \overline{Y_n^m(\theta,\emptyset)}, 0 \leq m \leq n$$

type 2: https://www.physics.uoguelph.ca/chapter-4-spherical-harmonics

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \qquad m \geq 0$$

↓

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$$Y_n^m(\theta,\emptyset) = \begin{cases} (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(cos\theta) e^{im\emptyset} & ,0 \leq m \leq n \\ \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(cos\theta) e^{im\emptyset} & ,-n \leq m < 0 \end{cases}$$

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type 3:

$$Y_n^m(\theta,\emptyset) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(cos\theta) e^{im\emptyset}, -n \leq m \leq n$$

$$Y_n^{-m}(\theta,\emptyset) = \overline{Y_n^m(\theta,\emptyset)}, -n \leq m \leq n$$

Why FMM define Y_n^m using type 3? (simple)

type 1:

$$Y_n^m(\theta,\emptyset)=\begin{cases}\sqrt{\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta)e^{im\emptyset} & ,0\le m\le n\\ (-1)^m\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(cos\theta)e^{im\emptyset} & ,-n\le m<0\end{cases}$$

type 2:

$$Y_n^m(\theta,\emptyset)=\begin{cases}(-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta)e^{im\emptyset} & ,0\le m\le n\\ \sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(cos\theta)e^{im\emptyset} & ,-n\le m<0\end{cases}$$

spherical harmonic addition theorem:

$$P_n(cos\gamma)=\sum_{m=-n}^{n}Y_n^m(\theta_1,\emptyset_1)\overline{Y_n^m(\theta_2,\emptyset_2)}$$

$$=\sum_{m=-n}^{n}Y_n^m(\theta_1,\emptyset_1)(-1)^mY_n^{-m}(\theta_1,\emptyset_1)$$

redefine

A New Version of the Fast Multipole Method for the Laplace Equation in Three Dimensions.

$$P_n(\cos\gamma)=\sum_{m=-n}^{n}Y_n^{-m}(\alpha,\beta)Y_n^m(\theta,\phi).$$

$$Y_n^{-m}(\theta,\emptyset)=\overline{Y_n^m(\theta,\emptyset)}$$

type 3:

$$Y_n^m(\theta,\phi)=\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\cdot P_n^{|m|}(\cos\theta)e^{im\phi}.$$

Why FMM define Y_n^m using type 3? (detailed)

type 1:

$$Y_n^m(\theta,\emptyset) = \begin{cases} \sqrt{\frac{(n-m)!}{(n+m)!}}\, P_n^m(cos\theta)e^{im\emptyset} & ,0 \le m \le n \\ (-1)^m \sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(cos\theta)e^{im\emptyset} & ,-n \le m < 0 \end{cases}$$

type 2:

$$Y_n^m(\theta,\emptyset) = \begin{cases} (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}}\, P_n^m(cos\theta)e^{im\emptyset} & ,0 \le m \le n \\ \sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(cos\theta)e^{im\emptyset} & ,-n \le m < 0 \end{cases}$$

spherical harmonic addition theorem:

$$\begin{aligned} P_n(cos\gamma) &= \sum_{m=-n}^{n} Y_n^m(\theta_1,\emptyset_1)\overline{Y_n^m(\theta_2,\emptyset_2)} \\ &= \sum_{m=0}^{n} Y_n^m(\theta_1,\emptyset_1)\overline{Y_n^m(\theta_2,\emptyset_2)} + \sum_{m=1}^{n} Y_n^{-m}(\theta_1,\emptyset_1)\overline{Y_n^{-m}(\theta_2,\emptyset_2)} \\ &= \sum_{m=0}^{n} Y_n^m(\theta_1,\emptyset_1)\overline{Y_n^m(\theta_2,\emptyset_2)} + \sum_{m=1}^{n} (-1)^m\overline{Y_n^m(\theta_1,\emptyset_1)} \cdot \overline{(-1)^m\overline{Y_n^m(\theta_2,\emptyset_2)}} \\ &= \sum_{m=0}^{n} Y_n^m(\theta_1,\emptyset_1)\overline{Y_n^m(\theta_2,\emptyset_2)} + \sum_{m=1}^{n} \overline{Y_n^m(\theta_1,\emptyset_1)}\, Y_n^m(\theta_2,\emptyset_2) \end{aligned}$$

redefine：设法让$m \le 0$时实际算出来的是原始Y_n^m的共轭

$$Y_n^m(\theta,\emptyset) = \begin{cases} Y_n^m(\theta,\emptyset) & m \ge 0 \\ \overline{Y_n^{|m|}(\theta,\emptyset)} & m < 0 \end{cases}$$

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$$Y_n^{-m}(\theta,\emptyset) = \overline{Y_n^m(\theta,\emptyset)}$$

type 3:

$$Y_n^m(\theta,\phi) = \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \cdot P_n^{|m|}(\cos\theta)e^{im\phi}.$$